Problem-Solving Strategies for continuous charge distributions

Problem-Solving Strategies Summing Electric Fields

We have discussed how electric field can be calculated for both the discrete and continuous charge distributions.
For the former, we apply the superposition principle:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \ .$$

For the latter, we must evaluate the vector integral

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r^2} \hat{\mathbf{r}} \ ,$$

where r is the distance from dq to the field point P and r^ is the corresponding unit vector.
To complete the integration, we shall follow the procedures outlined below:

(1) Start with $d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2} \hat{\mathbf{r}}$.

(2) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda\, d\ell & \text{(length)} \\ \sigma\, dA & \text{(area)} \\ \rho\, dV & \text{(volume)} \end{cases} ,$$

depending on whether the charge is distributed over a length, an area, or a volume.

Problem-Solving Strategies

(3) Substitute dq into the expression for $d\vec{\mathbf{E}}$.

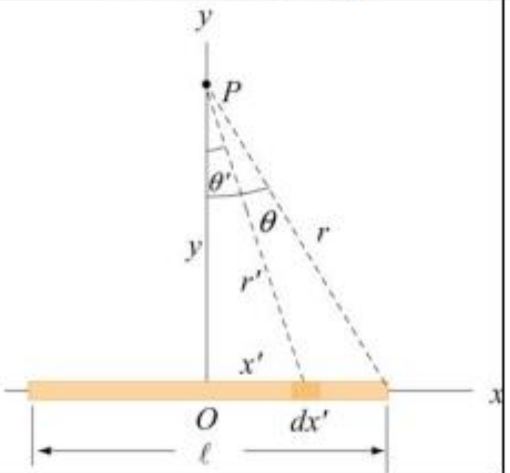
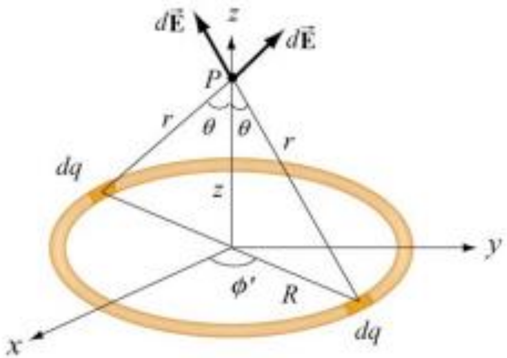
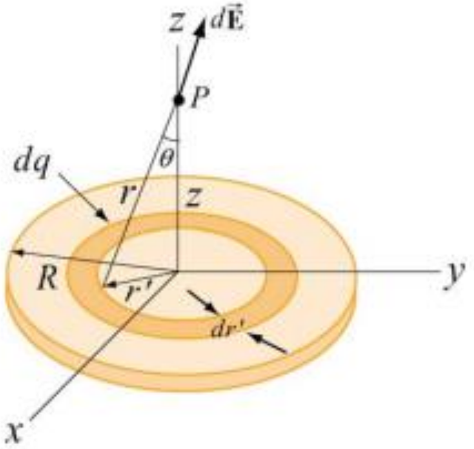
(4) Specify an appropriate coordinate system (Cartesian, cylindrical or spherical) and express the differential element ($d\ell$, dA, or dV) and r in terms of the coordinates (see Table 2.1 below for summary.)

	Cartesian (x, y, z)	Cylindrical (ρ, ϕ, z)	Spherical (r, θ, ϕ)
dl	$dx,\ dy,\ dz$	$d\rho,\ \rho d\phi,\ dz$	$dr,\ r\,d\theta,\ r\sin\theta\,d\phi$
dA	$dx\,dy,\ dy\,dz,\ dz\,dx$	$d\rho\,dz,\ \rho\,d\phi\,dz,\ \rho\,d\phi\,d\rho$	$r\,dr\,d\theta,\ r\sin\theta\,dr\,d\phi,\ r^2\sin\theta\,d\theta\,d\phi$
dV	$dx\,dy\,dz$	$\rho\,d\rho\,d\phi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\phi$

(5) Rewrite $d\vec{\mathbf{E}}$ in terms of the integration variable(s), and apply symmetry argument to identify non-vanishing component(s) of the electric field.

(6) Complete the integration to obtain $\vec{\mathbf{E}}$.

In the Table below we illustrate how the above methodologies can be utilized to compute the electric field for an infinite line charge, a ring of charge and a uniformly charged disk.

<table>
<tr><th></th><th>Line charge</th><th>Ring of charge</th><th>Uniformly charged disk</th></tr>
<tr><td>Figure</td><td>
</td><td>
</td><td>
</td></tr>
<tr><td>(2) Express dq in terms of charge density</td><td>$dq = \lambda\, dx'$</td><td>$dq = \lambda\, d\ell$</td><td>$dq = \sigma\, dA$</td></tr>
<tr><td>(3) Write down dE</td><td>$dE = k_e \frac{\lambda\, dx'}{r'^2}$</td><td>$dE = k_e \frac{\lambda\, dl}{r^2}$</td><td>$dE = k_e \frac{\sigma\, dA}{r^2}$</td></tr>
</table>

(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $\cos\theta = \dfrac{y}{r'}$ $r' = \sqrt{x'^2 + y^2}$	$d\ell = R\,d\phi'$ $\cos\theta = \dfrac{z}{r}$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r'\,dr'$ $\cos\theta = \dfrac{z}{r}$ $r = \sqrt{r'^2 + z^2}$
(5) Apply symmetry argument to identify non-vanishing component(s) of dE	$dE_y = dE\cos\theta$ $= k_e \dfrac{\lambda y\,dx'}{(x'^2 + y^2)^{3/2}}$	$dE_z = dE\cos\theta$ $= k_e \dfrac{\lambda Rz\,d\phi'}{(R^2 + z^2)^{3/2}}$	$dE_z = dE\cos\theta$ $= k_e \dfrac{2\pi\sigma z r'\,dr'}{(r'^2 + z^2)^{3/2}}$
(6) Integrate to get E	$E_y = k_e \lambda y \int_{-\ell/2}^{+\ell/2} \dfrac{dx}{(x^2 + y^2)^{3/2}}$ $= \dfrac{2k_e\lambda}{y} \dfrac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = k_e \dfrac{R\lambda z}{(R^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi'$ $= k_e \dfrac{(2\pi R\lambda)z}{(R^2 + z^2)^{3/2}}$ $= k_e \dfrac{Qz}{(R^2 + z^2)^{3/2}}$	$E_z = 2\pi\sigma k_e z \int_0^R \dfrac{r'\,dr'}{(r'^2 + z^2)^{3/2}}$ $= 2\pi\sigma k_e \left(\dfrac{z}{\lvert z \rvert} - \dfrac{z}{\sqrt{z^2 + R^2}} \right)$

Problem-Solving Strategies using Gauss' Law

1. Select a Gaussian surface with symmetry that "matches" the charge distribution.

   Use symmetry to determine the direction of $\vec{E}$ on the Gaussian surface.

   You want $\vec{E}$ to be constant in magnitude and everywhere perpendicular to the surface, so that $\vec{E} \cdot d\vec{A} = E\, dA$ …

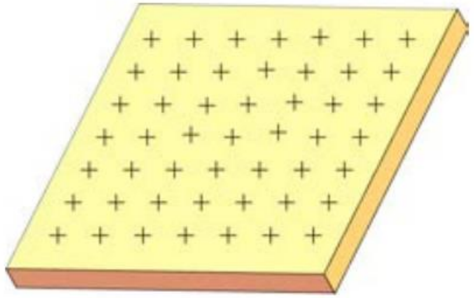
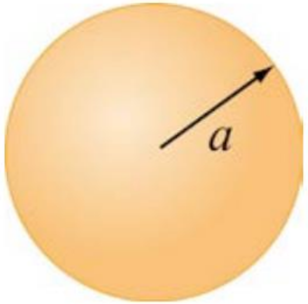
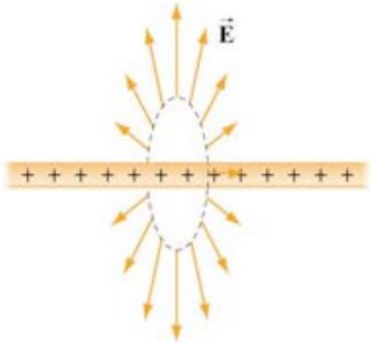
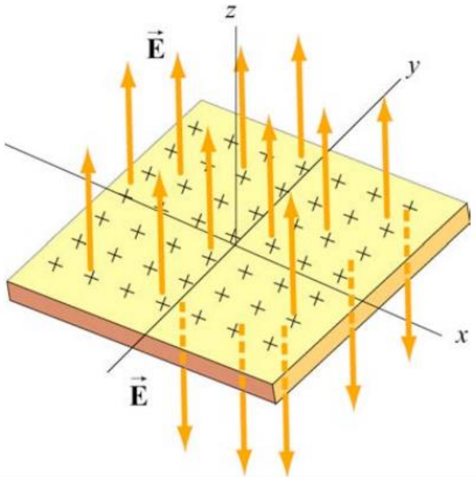
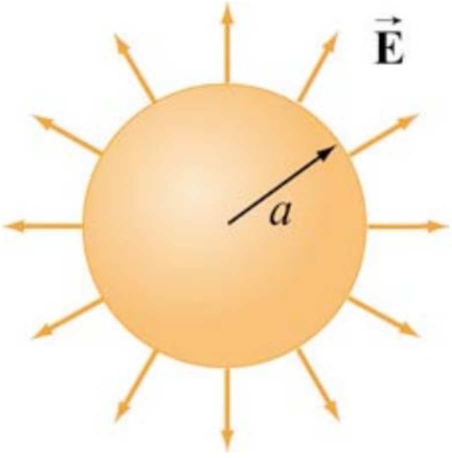
   … or else everywhere parallel to the surface so that $\vec{E} \cdot d\vec{A} = 0$.

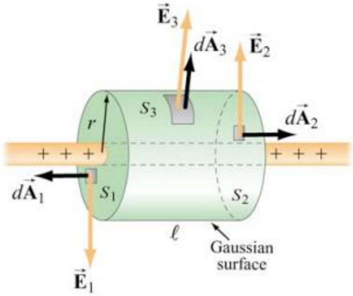
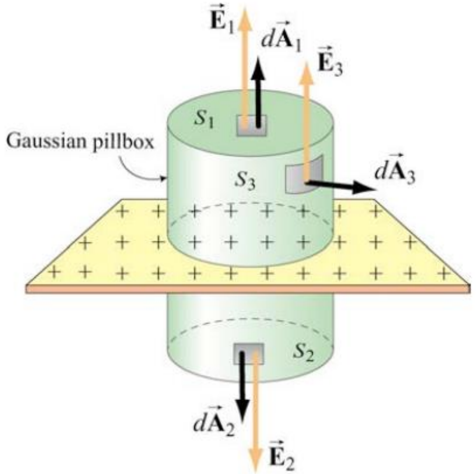
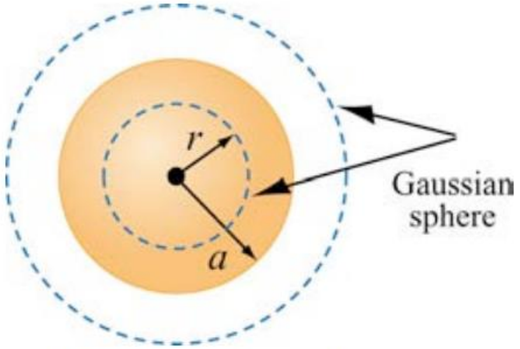
2. Evaluate the surface integral (electric flux).

3. Determine the charge inside the Gaussian surface.

4. Solve for $\vec{E}$.

Don't forget that to completely specify a vector, your answer must contain information about its direction.

System	**Infinite line of charge**	**Infinite plane of charge**	**Uniformly charged solid sphere**
			
Identify the symmetry	Cylindrical	Planar	Spherical
Determine the direction of $\vec{\mathbf{E}}$			
Divide the space into different regions	$r > 0$	$z > 0$ and $z < 0$	$r \le a$ and $r \ge a$

Choose Gaussian surface	 Coaxial cylinder	 Gaussian pillbox	 Concentric sphere
Calculate electric flux	$\Phi_E = E(2\pi r l)$	$\Phi_E = EA + EA = 2EA$	$\Phi_E = E(4\pi r^2)$
Calculate enclosed charge q_{in}	$q_{\text{enc}} = \lambda l$	$q_{\text{enc}} = \sigma A$	$q_{\text{enc}} = \begin{cases} Q(r/a)^3 & r \le a \\ Q & r \ge a \end{cases}$
Apply Gauss's law $\Phi_E = q_{\text{in}} / \varepsilon_0$ to find E	$E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$	$E = \dfrac{\sigma}{2\varepsilon_0}$	$E = \begin{cases} \dfrac{Qr}{4\pi\varepsilon_0 a^3}, & r \le a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r \ge a \end{cases}$